# Math 347 Worksheet

## Worksheet 10: Binomial Theorem and Binomial Coefficients

October 31, 2018

1) Use the binomial theorem to prove that $|P(S)| = 2^{|S|}$ for a finite set $S$.

2) Prove the following identities about binomial coefficients:

   (i) Basic identity
   $$\binom{n}{k} = \binom{n}{n-k};$$

   (ii) Pascal's identity, for all $0 \leq k \leq n$
   $$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1};$$

   (iii) Chairperson identity
   $$k\binom{n}{k} = n\binom{n-1}{k-1};$$

   (iv) Summation identity
   $$\sum_{i=0}^{n} \binom{i}{k} = \binom{n+1}{k+1}.$$

3) Calculate the number of non-negative integer solutions of $x_1 + x_2 + x_3 + x_4 = m$. What about the equation $x_1 + \cdots + x_k = n$?

4) Suppose that $n! + m! = k!$ for some $n, m, k \in \mathbb{N}$. Prove that $n = m = 1$ and $k = 2$.

5) By using a counting argument, prove that
   $$\binom{n}{k}\binom{k}{j} = \binom{n}{j}\binom{n-j}{k-j}.$$

6) A proof that $\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$ without induction.

   (a) Prove that
   $$i^2 = 2\binom{i}{2} + i.$$

   (b) Use the above result to find and prove the formula above.

7) What other summation formulas can you prove using the trick from Question 6)?